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DISSIPATION FUNCTIONS AND  
INVARIANT IMBEDDING—I

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### SUMMARY

In a series of papers dating from 1956, we have used the theory of invariant imbedding to derive a variety of nonlinear functional equations for the description of radiative transfer, neutron transport, diffusion and heat conduction, scattering and random walk, and wave propagation. In this paper we wish to present a new method for establishing the existence of solutions of these equations in the cases where no creation of matter is involved. This method is based upon the introduction of a new class of physically important functions, the dissipation functions. Combining these new functions with the functions we have repeatedly utilized in the past, the reflection and transmission functions, we easily obtain a basic conservation relation upon which all else hinges. The uniqueness proofs follow conventional lines.

## DISSIPATION FUNCTIONS AND INVARIANT IMBEDDING—I

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1. Introduction. In a series of papers dating from 1956 (see <sup>1</sup> for a complete bibliography), we have used the theory of invariant imbedding to derive a variety of nonlinear functional equations for the description of radiative transfer, neutron transport, diffusion and heat conduction, scattering and random walk, and wave propagation. In this paper we wish to present a new method for establishing the existence of solutions of these equations in the cases where no creation of matter is involved. This method is based upon the introduction of a new class of physically important functions, the dissipation functions. Combining these new functions with the functions we have repeatedly utilized in the past, the reflection and transmission functions, we easily obtain a basic conservation relation upon which all else hinges. The uniqueness proofs follow conventional lines.

It is, of course, essential for the rigorous formulation of the invariant imbedding approach to present these existence theorems. The results we present are, however, of deeper significance than mere rigor. In the first place, upon them we can found a new approach to the classical linear transport processes, one that is independent of characteristic values and spectral theory. In the second place, as Chandrasekhar<sup>2</sup> points out, invariance principles cannot always be guaranteed to yield a unique description of a physical process. Consequently, it is essential for computational purposes to know that we have a guaranteed method of obtaining the physically meaningful solutions.

2. The Mathematical Model. We wish to consider a particle process involving  $N$  types of particles traversing a homogeneous "rod" of length  $x$ . We suppose that when a particle of type  $j$  penetrates from either direction a segment of length  $\Delta$ , then

$b_{1j}\Delta$  incremental particles of type 1 are sent back in the opposite direction,  $b_{1j} \geq 0$ ,  $f_{11}\Delta$  particles disappear from the process, i.e., are "absorbed," and  $d_{1j}\Delta$  additional particles of type 1,  $1 \neq j$ , are sent through in the same direction. To conserve matter,  $d_{jj}\Delta \leq 0$  particles of type  $j$  are added to the original stream, where

$$d_{jj} = - \left\{ \sum_1 b_{1j} + \sum_{1 \neq j} d_{1j} + f_{jj} \right\} \leq 0, \quad j = 1, 2, \dots, N. \quad (1)$$

Here  $\Delta$  is an infinitesimal and all expressions are to terms in  $o(\Delta)$ .

We wish to determine the number of particles in state 1 which are reflected, transmitted and absorbed per unit time, when one particle of type  $j$  per unit time is incident upon the rod. Denote these quantities by  $r_{1j}(x)$ ,  $t_{1j}(x)$ , and  $\ell_{1j}(x)$  respectively, and introduce the  $N \times N$  matrices  $R(x) = (r_{1j}(x))$ ,  $T(x) = (t_{1j}(x))$ ,  $L(x) = (\ell_{1j}(x))$ ,  $B = (b_{1j})$ ,  $D = (d_{1j})$ , and the diagonal matrix  $F = (f_{11}\delta_{1j})$ .

Using techniques we have repeatedly applied, cf. <sup>1,3</sup>, we obtain the system of Riccati-type differential equations

$$\begin{aligned} R'(x) &= T(x)BT(x), \quad R(0) = 0, \\ T'(x) &= (D + R(x)B)T(x), \quad T(0) = I, \\ L'(x) &= FT(x) + L(x)BT(x), \quad L(0) = 0. \end{aligned} \quad (2)$$

It is this system of equations which we wish to discuss for all  $x > 0$ .

3. Physical Interpretation. The differentiation of particles can be made on the basis of type, energy, wave length, angular direction, and so on. The "rod" can represent a one-dimensional rod, a slab, or generally a collection of strata, as in <sup>4</sup>. The foregoing is thus a model of radiative transfer, multiple scattering, and similar processes.

4. Conservation and Existence Theorem. Our principal result is the following.

Theorem. Let

$$b_{1j} \geq 0, \quad d_{1j} \geq 0, \quad 1 \neq j, \quad f_{11} \geq 0, \quad (1)$$

$$M(B + D + F) = 0, \quad (2)$$

where M is the N x N matrix whose first row is composed of ones and whose other elements are zeroes.

Then:

- (a) The solution of (2.1) exists for all  $x \geq 0$  and is unique.
- (b)  $R(x)$ ,  $T(x)$  and  $L(x)$  are nonnegative for all  $x \geq 0$ .
- (c)  $M(R(x) + T(x) + L(x)) = M$  for  $x \geq 0$ . (3)
- (d) The limits  $R(\infty)$ ,  $T(\infty)$ ,  $L(\infty)$  exist.

The relation in (3.c) is the essential conservation relation.

5. Sketch of Proof. It is easy to show that solutions exist in a small  $x$ -interval,  $0 \leq x \leq x_0$ , satisfying the conditions  $R(x)$ ,  $T(x)$ ,  $L(x) \geq 0$ . For example, we can use the recurrence relations

$$\begin{aligned} R_{n+1}(x) &= \int_0^x T_n(y) B T_n(y) dy, \quad x \geq 0, \\ T_{n+1}(x) &= e^{Dx} + \int_0^x e^{D(x-y)} R_n(y) B T_n(y) dy, \end{aligned} \quad (1)$$

with  $R_0(x) = 0$ ,  $T_0(x) = I$ . The fact that  $T_{n+1}(x) \geq 0$  if  $R_n(y)$ ,  $T_n(y) \geq 0$  follows from the known result that  $e^{Dx} \geq 0$  if  $d_{1j} \geq 0$ ,  $1 \neq j$ ; see <sup>5</sup>.

If we now establish the conservation relation of (4.3c), which asserts the uniform boundedness of  $r_{1j}(x)$ ,  $t_{1j}(x)$ ,  $\ell_{1j}(x)$  over the interval  $[0, x_0]$ , then standard techniques in the theory of nonlinear differential equations will permit us to assert the existence of the solution for all  $x \geq 0$ . As mentioned above, uniqueness follows readily.

We have, using the equations of (2.1),

$$[M(R + T + L)]' = M(R + T + L)BT + M(D + F)T. \quad (2)$$

Consider this as a differential equation for the matrix

$Z = M(R + T + L)$ . It is satisfied over  $[0, x_0]$ , the interval of existence of  $R(x)$ ,  $T(x)$ ,  $L(x)$ , by the constant matrix  $Z = M$  since  $M(R(0) + T(0) + L(0)) = MI = M$  and

$$0 = M' = MBT + M(D + F)T = M(B + D + F)T = 0, \quad (3)$$

upon referring to (4.2), we see that (4.3c) holds over  $0 \leq x \leq x_0$ .

6. Discussion. In this paper, we have applied a new technique for establishing existence of solutions, based upon conservation principles, to one class of problems. In forthcoming papers, we will discuss more complex problems involving time-dependence, particle-particle interaction and variable media. We shall also discuss uses in transmission line theory and wave propagation.

1. R. Bellman, R. Kalaba, and G. M. Wing, "Invariant imbedding and mathematical physics—I: particle processes," J. Math. Physics, to appear.
2. S. Chandrasekhar, Radiative Transfer, Dover Publ., 1960, p. 190, et. seq.
3. R. Bellman, R. Kalaba, and G. M. Wing, "On the principle of invariant imbedding and neutron transport theory—I: one-dimensional case," J. Math. and Mech., vol. 7, 1958, pp. 149-162.
4. —————, "Invariant imbedding and neutron transport theory—IV: generalized transport theory," J. Math. and Mech., vol. 8, 1959, pp. 575-584.
5. R. Bellman, Introduction to Matrix Analysis, McGraw-Hill Book Co., Inc., New York, 1960, Theorem 3, p. 170.